Variational Method for Design Sensitivity Analysis in Nonlinear Structural Mechanics

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This paper describes a unified variational theory of design sensitivity analysis of linear and nonlinear structures for shape, nonshape, and material selection problems. The concepts of reference volume and adjoint structure are used to develop the unified viewpoint. A general formula for design sensitivity analysis is derived. Several analytical linear and nonlinear examples are used to interpret various terms of the formula and demonstrate its use. Such analytical examples give insight into the application of the general formula to more complex problems that must be treated numerically. Adjoint state fields are also interpreted as sensitivities for the functional related to the primary structure.

I. Introduction

DESIGN sensitivity analysis gives trend information that can be used in the conventional or optimal design process. The subject, therefore, has received considerable attention in recent years. For a thorough review of the subject, Refs. 1 and 2 should be consulted.

The present paper describes a unified variational theory of design sensitivity analysis of linear and nonlinear structures (geometric as well as physical nonlinearities) for shape, nonshape, and material selection problems. Time-dependent materials and cyclic plasticity are not considered. The adjoint structure concept is utilized to develop the design sensitivity expression. In Sec. II, equations of continuum mechanics for nonlinear analysis are summarized. They are needed in design sensitivity analysis. A unified viewpoint for shape and nonshape design sensitivity analysis is described in Sec. III. The concept of a reference volume is explained in Sec. IV. The variational theory of design sensitivity analysis using the adjoint variable approach is developed in Sec. V. The theory is used to solve four analytical problems in Sec. VI. Finally, concluding remarks are given in Sec. VII.

II. Nonlinear Analysis

Nonlinearities in structural systems can be due to large displacements, large strains, material behavior, and boundary conditions. Consistent theories to treat these nonlinearities have been developed.[3,4] We shall use the developments and notations of Ref. 4 and follow the total Lagrangian (or Lagrangian) formulation, although updated Lagrangian formulation can also be used. One of the major difficulties in describing nonlinear analysis is the complexity of notation. For the most part, we shall use matrix and tensor notations and standard symbols from the literature for various quantities. One major departure from linear analysis is that quantities must be measured in a deformed configuration. Also, a reference configuration for the quantities must be defined and identified. We shall use a left superscript to indicate the configuration in which the quantity occurs and a left subscript to indicate the reference configuration.

A starting point for the theory of nonlinear analysis is the principle of virtual work for the body in the deformed configuration at time t (load level t)

$$\int_{^0V} {}^t_0\boldsymbol{S} \cdot \delta^t_0\boldsymbol{\epsilon}^0 \, \mathrm{d}V - \int_{^0V} {}^t_0\boldsymbol{f} \cdot \delta^t\boldsymbol{u}^0 \, \mathrm{d}V - \int_{^0\Gamma_T} {}^t_0\boldsymbol{T}^0 \cdot \delta^t\boldsymbol{u}^0 \, \mathrm{d}\Gamma_T = 0 \quad (1)$$

where the left subscript 0 refers to the undeformed configuration, the multidot the standard tensor product, δ the variation of the state fields, and

^0V = undeformed volume of the body

${}^t_0\boldsymbol{S}$ = second Piola-Kirchhoff stress tensor

${}^t_0\boldsymbol{\epsilon}$ = Green-Lagrange strain tensor

${}^t_0\boldsymbol{f}$ = body force per unit volume

${}^t\boldsymbol{u}$ = displacement field

${}^t_0\boldsymbol{T}^0$ = surface traction specified on part of the surface $^0\Gamma_T$

$^0\Gamma$ = surface of the body

Let $\boldsymbol{u}^0$ be the specified displacement on the part Γ_u of the surface. Since the variation of the prescribed displacements is zero, the domain of the last integral is changed from $^0\Gamma$ to $^0\Gamma_T$. The variations of the state fields in Eq. (1) are arbitrary but kinematically admissible. The virtual work equation can also be written using Cauchy stress tensor and other quantities referred to the deformed configuration. Transformation can be used to recover Cauchy stresses from second Piola-Kirchhoff stresses and vice versa. However, in all the derivations given in this paper, we shall use the undeformed configuration as the reference configuration.

The Green-Lagrange strain tensor is given as

$${}^t_0\boldsymbol{\epsilon} = \tfrac{1}{2}\,[{}_0\nabla\,{}^t\boldsymbol{u}^T + ({}_0\nabla\,{}^t\boldsymbol{u}^T)^T + ({}_0\nabla\,{}^t\boldsymbol{u}^T)\,({}_0\nabla\,{}^t\boldsymbol{u}^T)^T] \quad (2)$$

The nonlinear stress-strain law, in general, can be written as

$${}^t_0\boldsymbol{S} = \phi({}^t_0\boldsymbol{\epsilon}, \boldsymbol{b}) \quad (3)$$

where $\boldsymbol{b}$ is the design variable vector. Note that for many applications, the functional form for ϕ is not known. In numerical implementations, the explicit form is not needed. Only an incremental stress-strain relation is required. For time-dependent materials, ϕ takes an integral form.

Equations (1-3) are nonlinear in the displacement field ${}^t\boldsymbol{u}$. There are several methods for solving such a system of equations.[5] The incremental/iterative procedure based on

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Newton methods is the most commonly used and effective procedure; it will be summarized here. In the derivation of the procedure, it is assumed that the equilibrium is known at t and it is desired at $t+\Delta t$. The state fields are decomposed as[4]

$$^{t+\Delta t}u = {}^t u + u; \qquad ^{t+\Delta t}_{0}S = {}^t_0 S + {}_0 S; \qquad {}_0S = \phi_{,\epsilon}\cdot {}_0\epsilon$$

$$^{t+\Delta t}_{0}\epsilon = {}^t_0\epsilon + {}_0\epsilon; \qquad ^{t+\Delta t}_{0}f = {}^t_0 f + {}_0 f; \qquad ^{t+\Delta t}_{0}T^0 = {}^t_0 T^0 + {}_0T^0 \tag{4}$$

where

u = increment in the displacement field

${}_0S$ = increment in the second Piola-Kirchhoff stress

${}_0\epsilon$ = increment in the Green-Lagrange strain

${}_0f$ = increment in the body force

${}_0T^0$ = increment in the surface traction

Variation of the strain field is given as

$$\delta {}^{t+\Delta t}_{0}\epsilon = \delta_0\epsilon \tag{5}$$

The incremental strain field from Eq. (2) is given as

$${}_0\epsilon = {}_0e + {}_0\eta \tag{6}$$

$${}_0e = \tfrac{1}{2}\,[{}_0\nabla u^T + ({}_0\nabla u^T)^T + ({}_0\nabla u^T)\,({}_0\nabla {}^t u^T)^T + ({}_0\nabla {}^t u^T)\,({}_0\nabla u^T)^T] \tag{7}$$

$${}_0\eta = \tfrac{1}{2}\,[({}_0\nabla u^T)\,({}_0\nabla u^T)^T] \tag{8}$$

If we substitute Eqs. (4–6) into Eq. (1), the virtual work principle at $t+\Delta t$ becomes

$$\int ({}^t_0S + {}_0S)\cdot \delta_0\epsilon^0\,\mathrm{d}V - \int {}^{t+\Delta t}_{0}f\cdot \delta u^0\,\mathrm{d}V - \int {}^{t+\Delta t}_{0}T^0\cdot \delta u^0\,\mathrm{d}\Gamma_T = 0 \tag{9}$$

Equation (9) is still nonlinear in incremental displacement field u. It is linearized by assuming

$$\delta_0\epsilon = \delta_0 e; \qquad {}_0S = \phi_{,\epsilon}\cdot {}_0e \tag{10}$$

and iteration is used within the load increment to satisfy the equilibrium exactly at $t+\Delta t$. The finite-element procedure has been used to implement the preceding equations into a computer program, ADINA.[6]

III. Unification of Dimensional and Shape Design Sensitivity Analyses

In the literature, shape and dimensional design sensitivity analysis problems have been treated independently. In the shape problem, the domain is allowed to vary, whereas in the dimensional problem, it is fixed but cross-sectional dimensions are allowed to vary. It will be seen here that when variational formulation with volume integrals is used, there is no distinction between the two problems.

Consider the general functional requiring design sensitivity analysis

$$\psi = \int_{{}^0V(b)} \bar{G}({}^t_0S,\, {}^t_0\epsilon,\, {}^tu, b)^0\,\mathrm{d}V + \int_{{}^0\Gamma_u(b)} \bar{g}({}^t_0T, b)^0\,\mathrm{d}\Gamma_u + \int_{{}^0\Gamma_T(b)} \bar{h}({}^tu, b)^0\,\mathrm{d}\Gamma_T \tag{11}$$

where ${}^0\Gamma_u$ is that part of the surface ${}^0\Gamma$ where the displacements are prescribed. It can be seen that when design b is changed, the volume of the body as well as its surface changes. As examples, consider the optimal design of two simple bodies shown in Fig. 1. Are these shape or dimensional optimization

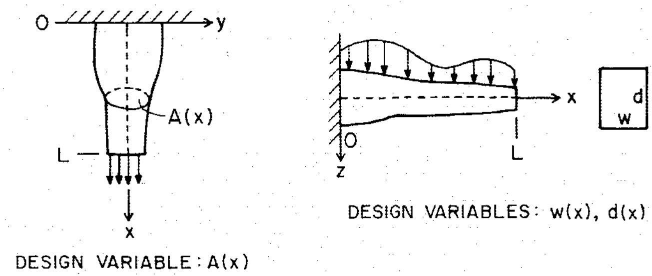

Fig. 1 Examples of optimal design.

problems? Our contention is that although the length of the members is not treated as a design variable in these problems, the volume of the body changes whenever any of the indicated design variables changes. We must account for variations of the domain of the body while writing design variations of the functional ψ in Eq. (11). Thus, the variational concept for design sensitivity analysis is slightly different from the corresponding concept used in pure analysis problems where the domain of the body is usually assumed to remain fixed. This distinction is important in maintaining the generality of the variational design sensitivity analysis theory in which the variation of the domain should always be considered.

IV. Concept of Reference Volume

The concept of a reference volume is extremely useful in problems where the volume of the body is changing. The idea, introduced recently in Ref. 7 for shape optimization problems, is to map the volume of the body in various configurations to a reference volume $\bar{V}$ as shown in Fig. 2. The original volume of the body ${}^0V(b^0)$ moves to a volume ${}^tV(b^0)$ under a nonlinear motion. However, both the volumes can be mapped to the fixed reference volume $\bar{V}$ under the mappings $F_1(b^0)$ and $F_2(b^0)$, respectively. The design process changes the shape of the body so that its volume becomes ${}^0V(b^1)$ at the new design b^1. This volume moves to ${}^tV(b^1)$ under the nonlinear motion. Both of these volumes can also be mapped to the fixed reference volume $\bar{V}$.

For design sensitivity analysis, all integrals of the problem are transformed to the reference volume using proper transformation of the independent variables. The mapping to the fixed volume keeps changing under state or design variations. However, the reference volume never changes. Thus, when

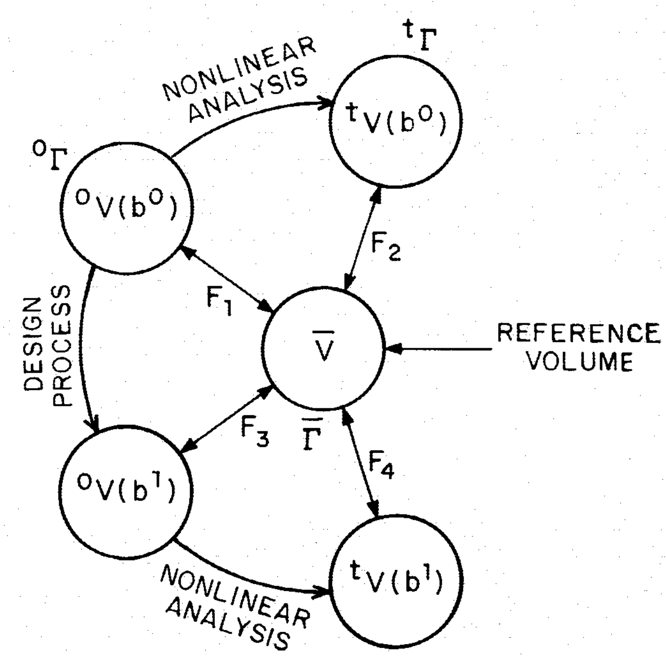

Fig. 2 Concept of reference volume.

variations of various integrals are taken, the variations of the reference volume need not be considered. In numerical implementations, this concept is also very useful. It allows us to discretize the design problem into design elements that keep fixed shape even when the real shape for the structure changes during the optimization process. If we use the transformation of independent variables, the various expressions are given as follows.

Virtual work Eq. (1) at load level t:

$$\int {}_0^tS \cdot \delta_0^t\epsilon J \, d\bar{V} - \int {}_0^tf \cdot \delta^t u J \, d\bar{V} - \int {}_0^tT^0 \cdot \delta^t u \, \bar{J} \, d\bar{\Gamma}_T = 0 \quad (12)$$

Incremental virtual work Eq. (9) at load level $t + \Delta t$:

$$\int ({}_0^tS + {}_0S) \cdot \delta_0\epsilon J \, d\bar{V} - \int {}^{t+\Delta t}_0f \cdot \delta u J \, d\bar{V} - \int {}^{t+\Delta t}_0T^0 \cdot \delta u \bar{J} \, d\bar{\Gamma}_T = 0 \quad (13)$$

Green-Lagrange strain tensor of Eq. (2):

$${}_0^t\epsilon = \tfrac{1}{2} [\bar{X}^T({}_r\nabla {}^tu^T) + ({}_r\nabla {}^tu^T)^T\bar{X} + \bar{X}^T({}_r\nabla {}^tu^T)({}_r\nabla {}^tu^T)^T\bar{X}] \quad (14)$$

Incremental strains of Eqs. (7) and (8):

$${}_0e = \tfrac{1}{2} [\bar{X}^T({}_r\nabla u^T) + ({}_r\nabla u^T)^T\bar{X} + \bar{X}^T({}_r\nabla u^T)({}_r\nabla {}^tu^T)^T\bar{X} + \bar{X}^T({}_r\nabla u^T)({}_r\nabla {}^tu^T)^T\bar{X}] \quad (15)$$

$${}_0\eta = \tfrac{1}{2} [\bar{X}^T({}_r\nabla u^T)({}_r\nabla u^T)^T\bar{X}] \quad (16)$$

Functional for sensitivity analysis of Eq. (11):

$$\psi = \int G({}_0^tS, {}_0^t\epsilon, {}^tu, b)J \, d\bar{V} + \int g({}_0^tT, b)\bar{J} \, d\bar{\Gamma}_u + \int h({}^tu, b)\bar{J} \, d\bar{\Gamma}_T \quad (17)$$

Jacobian of transformation:

$$X = \frac{\partial({}^0x, {}^0y, {}^0z)}{\partial({}^rx, {}^ry, {}^rz)}; \quad J = |X|; \quad \bar{X} = X^{-1}; \quad \bar{J} = J\|\bar{X}^Tn\| \quad (18)$$

In the foregoing equations, superscript or subscript r refers to the reference coordinates, $\bar{J}$ the area metric, and n the unit surface normal. Note that all quantities in the foregoing integrals are functions of the reference coordinates. Also, for oriented bodies such as bars and beams, J and $|X|$ may be different from each other if we use volume integrals throughout the sensitivity analysis. This can be observed in the examples discussed later.

Since the body is in equilibrium at load level t, the virtual work equation [Eq. (12)] is satisfied. This fact reduces the incremental equilibrium equation [Eq. (13)] as follows:

$$\int {}_0S \cdot \delta_0\epsilon J \, d\bar{V} + \int {}_0^tS \cdot \delta_0\eta J \, d\bar{V} - \int {}_0f \cdot \delta u J \, d\bar{V} - \int {}_0T^0 \cdot \delta u\bar{J} \, d\bar{\Gamma}_T = 0 \quad (19)$$

The fact that δ_0e is a kinematically admissible strain variation corresponding to δu (the variation of the incremental displacement) has been used to obtain Eq. (19). Equation (19) is still nonlinear in the increment u. It can be linearized using Eq. (10) as follows:

$$\int {}_0S \cdot \delta_0eJ \, d\bar{V} + \int {}_0^tS \cdot \delta_0\eta J \, d\bar{V} - \int {}_0f \cdot \delta u J \, d\bar{V} - \int {}_0T^0 \cdot \delta u\bar{J} \, d\bar{\Gamma}_T = 0 \quad (20)$$

Equation (20) will be used later to define the equilibrium equation for the adjoint structure.

V. Adjoint Structure Approach for General Design Sensitivity Analysis

The discrete form of the adjoint variable method has been discussed by several researchers.[1,8-13] The variational form of the approach is described in Ref. 13 where sensitivity with respect to shape variations is also considered using the material derivative idea. The adjoint structure approach is described in Refs. 14-17. The approach is applied to some nonlinear and shape variation problems in Refs. 18-20. Recently, Belegundu[21] has traced the roots of the adjoint variable method to methods of sensitivity analysis in optimal control literature. In addition, he has shown that sensitivity analysis methods for static, dynamic, shape, and distributed parameter problems can be viewed as the general Lagrange multiplier method. This generalization shows that the adjoint variable is also a Lagrange multiplier for the state equations that give a sensitivity interpretation for it.[22] This interpretation is extremely useful and leads to some insights into the adjoint variable method. It also has implications in practical applications and numerical implementations of the method.

Considerable numerical work has been done for the design sensitivity analysis and optimization of linear structures.[1] The material derivative approach has been exploited for shape optimization. In this regard, the recent work of Choi,[23] Yang and Botkin,[24] and Hou et al.[25] is significant. Yang and Botkin[24] have demonstrated the equivalence of variational and finite-element formulations of design sensitivity analysis of shape problems for linear structures. This equivalence can also be shown for nonlinear problems. Hou et al.[25] have discussed some difficulties with the material derivative approach of design sensitivity analysis of linear shape problems. They have suggested numerical procedures to improve the accuracy of the approach.

In the following derivation, we combine the adjoint structure approach with the fixed reference volume concept to develop a general theory of design sensitivity analysis of linear or nonlinear structures. To avoid confusion, we use δ and $\bar{\delta}$ to indicate arbitrary variations of the state fields and total variations with respect to the design variable, respectively. Further, to be more precise in some instances, we use $\tilde{\delta}$ and $\bar{\bar{\delta}}$ to represent implicit and explicit design variations of certain quantities; $\tilde{\delta}$ contains variations due to the implicit dependence of the state field. Also, the notation $G_{,S}$ will be used to indicate the partial derivative of G with respect to ${}_0^tS$, written in the reference coordinate system. Note that the virtual work equation (12) holds at the final deformed state of the system denoted by the left superscript t on various variables. The design sensitivity analysis is performed there.

Now taking the total variation of the functional ψ in Eq. (17) with respect to design, we obtain

$$\bar{\delta}\psi = \bar{\bar{\delta}}\psi + \tilde{\delta}\psi$$

where

$$\bar{\bar{\delta}}\psi = \int \bar{\bar{\delta}}GJ \, d\bar{V} + \int G\bar{\delta}J \, d\bar{V} + \int \bar{\bar{\delta}}g\bar{J} \, d\bar{\Gamma}_u + \int g\bar{\delta}\bar{J} \, d\bar{\Gamma}_u + \int h \, \bar{\delta}\bar{J} \, d\bar{\Gamma}_T + \int \bar{\bar{\delta}}h\bar{J} \, d\bar{\Gamma}_T \quad (21)$$

and

$$\tilde{\delta}\psi = \int (G_{,S} \cdot \bar{\delta}_0^tS + G_{,\epsilon} \cdot \bar{\delta}_0^t\epsilon + G_{,u} \cdot \bar{\delta}^tu)J \, d\bar{V} + \int g_{,T} \cdot \bar{\delta}_0^tT \, \bar{J} \, d\bar{\Gamma}_u + \int h_{,u} \cdot \bar{\delta}^tu\bar{J} \, d\bar{\Gamma}_T \quad (22)$$

Note that $\bar{\bar{\delta}}\psi$ contains variations of ψ involving explicit variation of J, $\bar{J}$, G, g, and h; and $\tilde{\delta}\psi$ contains variations of ψ involving state fields that depend implicitly as well as explicitly on the design variables—the explicit dependence comes through the transformation of the fields to a set of reference coordinates. The basic idea of the adjoint structure approach is to replace the implicit design variations of the state fields in

Eq. (22) by the explicit design variations and the adjoint state fields. To accomplish this, we shall define adjoint fields and write design variations of various equations.

The adjoint field shall be identified with a superscript a. Let the adjoint structure be defined as follows:

Initial adjoint strain field:

$$\epsilon^{ai} = G_{,s} \tag{23a}$$

Initial adjoint stress field:

$$S^{ai} = G_{,\epsilon} \tag{23b}$$

Adjoint body force:

$$f^a = G_{,u} \tag{23c}$$

Specified traction for adjoint structure:

$$T^{a0} = h_{,u} \text{ on the traction surface } {}^0\Gamma_T \tag{24a}$$

Specified displacements for adjoint structure:

$$u^{a0} = -g_{,T} \text{ on the displacement boundary } {}^0\Gamma_u \tag{24b}$$

Constitutive law (linear) for the adjoint structure:

$$S^a = \phi^T_{,\epsilon}\cdot(\epsilon^a - \epsilon^{ai}) - S^{ai}; \quad S^a = \text{ the adjoint stress field} \tag{24c}$$

Adjoint strain field [linear in u^a defined using Eq. (15)]:

$$\epsilon^a = \tfrac{1}{2}\,[\bar X^T({}_r\nabla u^{aT}) + ({}_r\nabla u^{aT})^T\bar X + \bar X^T({}_r\nabla u^{aT})\,({}_r\nabla{}^t u^T)^T\bar X$$
$$+ \bar X^T({}_r\nabla{}^t u^T)\,({}_r\nabla u^{aT})^T\bar X] \tag{25}$$

Note that all the adjoint fields are also written in the reference coordinate system. Substituting the adjoint fields from Eqs. (23) into Eq. (22), we can express the implicit variation of the functional ψ as follows:

$$\bar\delta\psi = \int(\epsilon^{ai}\cdot\bar\delta^t_0 S + S^{ai}\cdot\bar\delta^t_0\epsilon + f^a\cdot\bar\delta^t u)J\,d\bar V$$
$$-\int u^{a0}\cdot\bar\delta^t_0 T\,\bar J\,d\bar\Gamma_u + \int T^{a0}\cdot\bar\delta^t u\,\bar J\,d\bar\Gamma_T \tag{26}$$

To eliminate the implicit variations of the state fields from Eq. (26), we shall take design variations of various equations associated with the primary structure and introduce the equilibrium equation for the adjoint structure.

Design variations of the constitutive law [Eq. (3)] and strains of Eq. (14) give

$$\bar\delta^t_0 S = \phi_{,\epsilon}\cdot\bar\delta^t_0\epsilon + \bar{\bar\delta}\phi \tag{27}$$

$$\bar\delta^t_0\epsilon = \tilde\delta^t_0\epsilon + \bar{\bar\delta}^t_0\epsilon \tag{28}$$

where

$$\tilde\delta^t_0\epsilon = \tfrac{1}{2}\,[\bar X^T({}_r\nabla\bar\delta^t u^T) + ({}_r\nabla\bar\delta^t u^T)^T\bar X + \bar X^T({}_r\nabla\bar\delta^t u^T)\,({}_r\nabla{}^t u^T)^T\bar X$$
$$+ \bar X^T({}_r\nabla{}^t u^T)\,({}_r\nabla\bar\delta^t u^T)^T\bar X] \tag{29}$$

$$\bar{\bar\delta}^t_0\epsilon = \tfrac{1}{2}\,[\delta\bar X^T({}_r\nabla{}^t u^T) + ({}_r\nabla{}^t u^T)^T\delta\bar X + \delta\bar X^T({}_r\nabla{}^t u^T)\,({}_r\nabla{}^t u^T)^T\bar X$$
$$+ \bar X^T({}_r\nabla{}^t u^T)\,({}_r\nabla{}^t u^T)^T\delta\bar X] \tag{30}$$

Note that $\tilde\delta^t u = \bar\delta^t u$ has been used.

Substituting $\bar\delta^t_0 S$ from Eq. (27) into Eq. (26) and using Eq. (23), we get

$$\bar\delta\psi = \int[(\epsilon^a\cdot\phi_{,\epsilon} - S^a)\cdot\bar\delta^t_0\epsilon + \epsilon^{ai}\cdot\bar{\bar\delta}\phi + f^a\cdot\bar\delta^t u]J\,d\bar V$$
$$-\int u^{a0}\cdot\bar\delta^t_0 T\bar J\,d\bar\Gamma_u + \int T^{a0}\cdot\bar\delta^t u\bar J\,d\bar\Gamma_T \tag{31}$$

Comparing Eqs. (25) and (29), we observe that the adjoint displacements u^a and strains ϵ^a are compatible and can be used as virtual displacements and virtual strains for the primary structure. Therefore, the following equation holds for the primary structure:

$$\int {}^t_0 S\cdot\epsilon^a J\,d\bar V - \int {}^t_0 f\cdot u^a J\,d\bar V - \int {}^t_0 T\cdot u^a\bar J\,d\bar\Gamma = 0 \tag{32}$$

Note that the last integral in Eq. (32) is over the entire surface that includes the reactive traction forces on the displacement specified boundary. After an equilibrium configuration has been determined, the reactive forces are determined using equilibrium conditions at the surface. Then, the reaction forces can be included in Eq. (32) as long as the compatible displacement field is used.[26] While taking design variations of Eq. (32), we need design variations of adjoint strains ϵ^a that can be written using Eq. (25) as follows:

$$\bar\delta\epsilon^a = \bar{\bar\delta}\epsilon^a + \tilde\delta\epsilon^a \tag{33}$$

where

$$\tilde\delta\epsilon^a = \tilde\delta e^a + \tilde\delta\eta^a \tag{34}$$

$$\bar{\bar\delta}\epsilon^a = \tfrac{1}{2}\,[\delta\bar X^T({}_r\nabla u^{aT}) + ({}_r\nabla u^{aT})^T\delta\bar X + \delta\bar X^T({}_r\nabla u^{aT})({}_r\nabla{}^t u^T)^T\bar X$$
$$+ \bar X^T({}_r\nabla u^{aT})({}_r\nabla{}^t u^T)^T\delta\bar X + \delta\bar X^T({}_r\nabla{}^t u^T)({}_r\nabla u^{aT})^T\bar X$$
$$+ \bar X^T({}_r\nabla{}^t u^T)({}_r\nabla u^{aT})^T\delta\bar X] \tag{35}$$

$$\tilde\delta e^a = \tfrac{1}{2}\,[\bar X^T({}_r\nabla\bar\delta u^{aT}) + ({}_r\nabla\bar\delta u^{aT})^T\bar X + \bar X^T({}_r\nabla\bar\delta u^{aT})({}_r\nabla{}^t u^T)^T\bar X$$
$$+ \bar X^T({}_r\nabla{}^t u^T)({}_r\nabla\bar\delta u^{aT})^T\bar X] \tag{36}$$

$$\tilde\delta\eta^a = \tfrac{1}{2}\,[\bar X^T({}_r\nabla u^{aT})({}_r\nabla\bar\delta^t u^T)^T\bar X$$
$$+ \bar X^T({}_r\nabla\bar\delta^t u^T)({}_r\nabla u^{aT})^T\bar X] \tag{37}$$

The design variation of the equilibrium equation (32) for the primary structure gives

$$\int\bar\delta^t_0 S\cdot\epsilon^a J\,d\bar V + \int {}^t_0 S\cdot\epsilon^a\,\bar\delta J\,d\bar V - \int\bar\delta^t_0 f\cdot u^a J\,d\bar V$$
$$-\int {}^t_0 f\cdot u^a\,\bar\delta J\,d\bar V - \int\bar\delta^t_0 T\cdot u^a\bar J\,d\bar\Gamma - \int {}^t_0 T\cdot u^a\,\bar\delta\bar J\,d\bar\Gamma$$
$$+\int {}^t_0 S\cdot\bar{\bar\delta}\epsilon^a J\,d\bar V + \int {}^t_0 S\cdot\tilde\delta\eta^a J\,d\bar V + \int {}^t_0 S\cdot\tilde\delta e^a J\,d\bar V$$
$$-\int {}^t_0 f\cdot\bar\delta u^a J\,d\bar V - \int {}^t_0 T\cdot\bar\delta u^a\bar J\,d\bar\Gamma = 0 \tag{38}$$

Note that since $\bar\delta u^a$ and $\tilde\delta e^a$ are compatible virtual displacements and strains for the primary structure, the last three integrals in Eq. (38) drop out due to equilibrium. Also, the body force and the specified surface traction are assumed independent of the displacement field in Eq. (38). Substitute for $\bar\delta^t_0 S$ from Eq. (27) into Eq. (38) and rearrange

$$\int\epsilon^a\cdot\phi_{,\epsilon}\cdot\bar\delta^t_0\epsilon J\,d\bar V = -\int[{}^t_0 S\cdot\bar{\bar\delta}\epsilon^a + {}^t_0 S\cdot\tilde\delta\eta^a - \bar\delta^t_0 f\cdot u^a$$
$$+\epsilon^a\cdot\bar{\bar\delta}\phi]J\,d\bar V + \int({}^t_0 f\cdot u^a - {}^t_0 S\cdot\epsilon^a)\,\bar\delta J\,d\bar V$$
$$+\int(\bar\delta^t_0 T\cdot u^a\bar J + {}^t_0 T\cdot u^a\,\bar\delta\bar J)\,d\bar\Gamma \tag{39}$$

Substitute Eq. (39) into Eq. (31) to obtain

$$\bar\delta\psi = \int[\bar\delta^t_0 f\cdot u^a - (\epsilon^a - \epsilon^{ai})\cdot\bar{\bar\delta}\phi - S^a\cdot\bar\delta^t_0\epsilon - {}^t_0 S\cdot\bar{\bar\delta}\epsilon^a - {}^t_0 S\cdot\tilde\delta\eta^a$$
$$+ f^a\cdot\bar\delta^t u]J\,d\bar V + \int({}^t_0 f\cdot u^a - {}^t_0 S\cdot\epsilon^a)\,\bar\delta J\,d\bar V$$
$$+\int({}^t_0 T^0\cdot u^a\,\bar\delta\bar J + \bar\delta^t_0 T^0\cdot u^a\,\bar J)\,d\bar\Gamma_T + \int {}^t_0 T\cdot u^{a0}\,\bar\delta\bar J\,d\bar\Gamma_u$$

$$+\int T^{a0}\cdot\bar{\delta}^t u\,\bar{J}\,\mathrm{d}\bar{\Gamma}_T \tag{40}$$

where the domain $\bar{\Gamma}$ of the surface integral in Eq. (39) has been divided into the parts $\bar{\Gamma}_T$ and $\bar{\Gamma}_u$, and ${}^t_0\boldsymbol{T}^0$ are the specified tractions on $\bar{\Gamma}_T$ and $\boldsymbol{u}^a=\boldsymbol{u}^{a0}$ on $\bar{\Gamma}_u$. Substitute Eq. (28) into Eq. (40) and rearrange to obtain

$$\begin{aligned}\bar{\delta}\psi &= \int [\bar{\delta}^t_0\boldsymbol{f}\cdot\boldsymbol{u}^a-(\boldsymbol{\epsilon}^a-\boldsymbol{\epsilon}^{ai})\cdot\bar{\bar{\delta}}\boldsymbol{\phi}-\boldsymbol{S}^a\cdot\bar{\bar{\delta}}^t_0\boldsymbol{\epsilon}-{}^t_0\boldsymbol{S}\cdot\bar{\bar{\delta}}\boldsymbol{\epsilon}^a]J\,\mathrm{d}\bar{V}\\ &+\int ({}^t_0\boldsymbol{f}\cdot\boldsymbol{u}^a-{}^t_0\boldsymbol{S}\cdot\boldsymbol{\epsilon}^a)\,\bar{\delta}J\,\mathrm{d}\bar{V}\\ &+\int ({}^t_0\boldsymbol{T}^0\cdot\boldsymbol{u}^a\,\bar{\delta}\bar{J}+\bar{\delta}^t_0\boldsymbol{T}^0\cdot\boldsymbol{u}^a\bar{J})\,\mathrm{d}\bar{\Gamma}_T\\ &+\int {}^t_0\boldsymbol{T}\cdot\boldsymbol{u}^{a0}\,\bar{\delta}\bar{J}\,\mathrm{d}\bar{\Gamma}_u-[\int(\boldsymbol{S}^a\cdot\bar{\delta}^t_0\boldsymbol{\epsilon}+{}^t_0\boldsymbol{S}\cdot\bar{\delta}\boldsymbol{\eta}^a-\boldsymbol{f}^a\cdot\bar{\delta}^t\boldsymbol{u})J\,\mathrm{d}\bar{V}\\ &-\int \boldsymbol{T}^{a0}\cdot\bar{\delta}^t\boldsymbol{u}\bar{J}\,\mathrm{d}\bar{\Gamma}_T]\end{aligned} \tag{41}$$

Now, if the last four terms in Eq. (41) vanish, then we will have $\bar{\delta}\psi$ in terms of only explicit variations and adjoint fields. The adjoint fields, however, still have to be determined. This motivates the definition of the equilibrium equation for the adjoint structure. If we use the linearized equilibrium equation (20) for the primary structure, the equilibrium equation for the adjoint structure is identified as follows:

$$\int(\boldsymbol{S}^a\cdot\delta^t_0\boldsymbol{\epsilon}+{}^t_0\boldsymbol{S}\cdot\delta\boldsymbol{\eta}^a-\boldsymbol{f}^a\cdot\delta^t\boldsymbol{u})J\,\mathrm{d}\bar{V}-\int\boldsymbol{T}^{a0}\cdot\delta^t\boldsymbol{u}\bar{J}\,\mathrm{d}\bar{\Gamma}_T=0 \tag{42}$$

Since the $\bar{\delta}^t\boldsymbol{u}$ is a kinematically admissible variation of the primary displacement field, it can replace $\delta^t\boldsymbol{u}$ in Eq. (42). Thus, the last four terms in Eq. (41) drop out to give

$$\begin{aligned}\bar{\delta}\psi &= \int [\bar{\delta}^t_0\boldsymbol{f}\cdot\boldsymbol{u}^a-(\boldsymbol{\epsilon}^a-\boldsymbol{\epsilon}^{ai})\cdot\bar{\bar{\delta}}\boldsymbol{\phi}-\boldsymbol{S}^a\cdot\bar{\bar{\delta}}^t_0\boldsymbol{\epsilon}-{}^t_0\boldsymbol{S}\cdot\bar{\bar{\delta}}\boldsymbol{\epsilon}^a]J\,\mathrm{d}\bar{V}\\ &+\int ({}^t_0\boldsymbol{f}\cdot\boldsymbol{u}^a-{}^t_0\boldsymbol{S}\cdot\boldsymbol{\epsilon}^a)\,\bar{\delta}J\,\mathrm{d}\bar{V}\\ &+\int ({}^t_0\boldsymbol{T}^0\cdot\boldsymbol{u}^a\,\bar{\delta}\bar{J}+\bar{\delta}^t_0\boldsymbol{T}^0\cdot\boldsymbol{u}^a\,\bar{J})\,\mathrm{d}\bar{\Gamma}_T\\ &+\int {}^t_0\boldsymbol{T}\cdot\boldsymbol{u}^{a0}\,\bar{\delta}\bar{J}\,\mathrm{d}\bar{\Gamma}_u\end{aligned} \tag{43}$$

Substitute Eq. (43) into Eq. (21) to obtain the final sensitivity formula as

$$\begin{aligned}\bar{\delta}\psi &= \int [\bar{\delta}^t_0\boldsymbol{f}\cdot\boldsymbol{u}^a-(\boldsymbol{\epsilon}^a-\boldsymbol{\epsilon}^{ai})\cdot\bar{\bar{\delta}}\boldsymbol{\phi}-\boldsymbol{S}^a\cdot\bar{\bar{\delta}}^t_0\boldsymbol{\epsilon}-{}^t_0\boldsymbol{S}\cdot\bar{\bar{\delta}}\boldsymbol{\epsilon}^a\\ &+\bar{\bar{\delta}}G]J\,\mathrm{d}\bar{V}+\int ({}^t_0\boldsymbol{f}\cdot\boldsymbol{u}^a-{}^t_0\boldsymbol{S}\cdot\boldsymbol{\epsilon}^a+G)\bar{\delta}J\,\mathrm{d}\bar{V}\\ &+\int (\bar{\bar{\delta}}g\bar{J}+g\,\bar{\delta}\bar{J}+{}^t_0\boldsymbol{T}\cdot\boldsymbol{u}^{a0}\,\bar{\delta}\bar{J})\,\mathrm{d}\bar{\Gamma}_u\\ &+\int [(h+{}^t_0\boldsymbol{T}^0\cdot\boldsymbol{u}^a)\bar{\delta}\bar{J}+(\bar{\bar{\delta}}h+\bar{\delta}^t_0\boldsymbol{T}^0\cdot\boldsymbol{u}^a)\bar{J}]\,\mathrm{d}\bar{\Gamma}_T\end{aligned} \tag{44}$$

Equation (44) is an explicit design sensitivity formula for linear and nonlinear structures (geometric and material nonlinearities) and shape, nonshape, and material selection problems. It can be discretized for computer implementation.

Sensitivity Interpretation for Adjoint Fields

It is interesting to note that adjoint fields can be interpreted as sensitivities of the quantities related to the primary structure. These interpretations, revealed directly by Eq. (44), can provide further useful sensitivity information for gaining insights into the design optimization process. In addition, the interpretations can be invaluable in practical applications and numerical implementations. For example, the following terms in Eq. (44) involving adjoint fields can be directly used to determine implicit design variations of the functional ψ:

1) $\int\bar{\delta}^t_0\boldsymbol{f}\cdot\boldsymbol{u}^a\,J\,\mathrm{d}\bar{V}$; with respect to the body force,
2) $\int -(\boldsymbol{\epsilon}^a-\boldsymbol{\epsilon}^{ai})\cdot\bar{\bar{\delta}}\boldsymbol{\phi}\,J\,\mathrm{d}\bar{V}$; with respect to the constitutive law,
3) $\int\bar{\delta}^t_0\boldsymbol{T}^0\cdot\boldsymbol{u}^a\,\bar{J}\,\mathrm{d}\bar{\Gamma}_T$; with respect to the specified surface tractions.

The sensitivity interpretation for the adjoint displacements have been also discussed in Refs. 21 and 22. Some of the interpretations will be observed in the following examples.

VI. Example Problems

Several analytical linear and nonlinear examples are solved to show the use of Eq. (44) and interpretation of various terms. Although these examples are simple, they can be valuable in gaining insights into numerical implementation for larger complex problems. Also in using Eq. (44) in this section, we will use standard symbols σ for stress and ϵ for strain.

Example 1. Bar Under Self Weight

This example is taken from Ref. 7 where sensitivity of tip displacement with respect to length L is calculated. We will calculate sensitivities with respect to all parameters of the problem to demonstrate use of formula (44) for material, cross-sectional, and length variations. The problem definition and various transformations are shown in Fig. 3. Note that $\bar{A}$ is the reference area coordinate that has unit value, and the area coordinate in the original space is $a(=A)$. Small displacements and linear stress-strain law are assumed. The displacement field for the bar is given as $u(x)=fx(2L-x)/2E$ where f is the body force per unit volume. Thus, $u(L)=fL^2/2E$ and

$$\bar{\delta}u(L)=(L^2/2E)\bar{\delta}f+(fL/E)\bar{\delta}L+(0)\bar{\delta}A-(fL^2/2E^2)\bar{\delta}E \tag{45}$$

There are at least two interpretations of this problem and both can be treated using Eq. (44).

First Interpretation

In this case, Eq. (44) can be interpreted as a line integral with x as the only independent variable. The stress-strain law of Eq. (3) must be interpreted as force-strain law, as the structure is only a line element. Note that this must be done with the formulas given in Refs. 14, 16, 18, and 20 when variations with respect to the cross-sectional area are needed. While using Eq. (44), the tip displacement can be treated as a boundary term or the interior term. We will use the latter approach. The functional for sensitivity analysis is given as

$$\psi=\int_0^1 u(\xi)J^{-1}\bar{\delta}(\xi-1)J\,\mathrm{d}\xi;\qquad G=u(\xi)J^{-1}\bar{\delta}(\xi-1)$$

$$G_{,u}=J^{-1}\bar{\delta}(\xi-1) \tag{46}$$

where $\bar{\delta}(\xi-1)$ is the Dirac delta function. The primary and adjoint fields can be obtained as

$$\begin{aligned}&u(\xi)=fL^2\xi(2-\xi)/2E; && u^a(\xi)=L\xi/EA\\ &\epsilon(\xi)=fL^2(1-\xi)/E; && \epsilon^a(\xi)=L/EA\\ &\epsilon=\epsilon(\xi)J^{-1}=fL(1-\xi)/E; && \epsilon^a=\epsilon^a(\xi)J^{-1}=1/EA\\ &N=EA\epsilon=fAL(1-\xi); && N^a=EA\epsilon^a=1\end{aligned} \tag{47}$$

where N is the axial force and $\phi=EA\epsilon$. Equation (44) reduces to

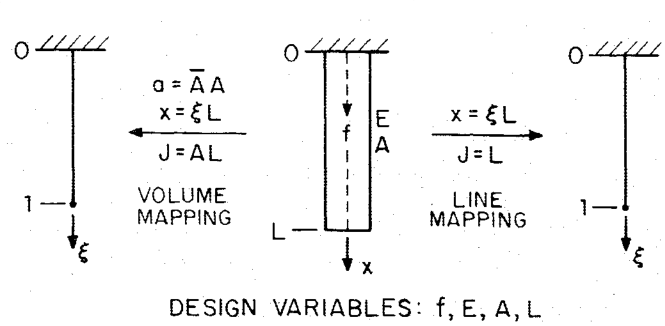

Fig. 3 Bar under self weight.

$$\bar{\delta}\psi = \int_0^1 (\bar{\bar{\delta}}\tilde{f}u^a - \epsilon^a \bar{\bar{\delta}}\phi - N^a \bar{\bar{\delta}}\epsilon - N\bar{\bar{\delta}}\epsilon^a + \bar{\bar{\delta}}G)J \, d\xi$$

$$+ \int_0^1 (\tilde{f}u^a - N\epsilon^a + G)\bar{\delta}J \, d\xi \tag{48}$$

Note that since we are using line integrals, the body force $\tilde{f} = fA$ must be used. Various quantities for use in Eq. (48) are

$$\bar{\bar{\delta}}\phi = (A\bar{\delta}E + E\bar{\delta}A)J^{-1}fL^2(1-\xi)/E$$

$$\bar{\bar{\delta}}\tilde{f} = A\bar{\bar{\delta}}f + f\bar{\delta}A; \; \bar{\bar{\delta}}G = -u(\xi)J^{-2}\hat{\delta}(\xi-1)\bar{\delta}L$$

$$\bar{\bar{\delta}}\epsilon = \epsilon(\xi)\bar{\delta}J^{-1} = -f(1-\xi)\bar{\delta}L/E$$

$$\bar{\bar{\delta}}\epsilon^a = \epsilon^a(\xi)\bar{\delta}J^{-1} = -\bar{\delta}L/EAL \tag{49}$$

Substituting all of the quantities in Eq. (48) and carrying out the integrations, we obtain the required sensitivity equation that is the same as Eq. (45).

It is noted that the terms $(L^2/2E)\bar{\delta}f$ and $(-fL^2/2E^2)\bar{\delta}E$ in Eq. (45) are obtained directly from the terms $\int \bar{\delta}_0^i f \cdot u^a J \, d\bar{V}$ and $\int -(\epsilon^a - \epsilon^{ai}) \cdot \bar{\bar{\delta}}\phi J \, d\bar{V}$, respectively, in Eq. (44) after proper substitutions and integrations. This is consistent with the sensitivity interpretations for the adjoint fields discussed in the previous section.

Second Intepretation

In this case, Eq. (3) will be treated as a volume integral. The functional for sensitivity analysis is given as

$$\psi = \int_0^1 \int_{\bar{A}} (AL)^{-1} u(\xi)\hat{\delta}(\xi-1)J \, d\bar{A} \, d\xi$$

$$G = (AL)^{-1}u(\xi)\hat{\delta}(\xi-1) \tag{50}$$

The displacement and strain fields are the same as given in Eq. (47). However, the stress-strain law is the usual Hooke's law

$$\sigma = E\epsilon = fL(1-\xi); \qquad \sigma^a = 1/A \tag{51}$$

Equation (44) reduces to

$$\bar{\delta}\psi = \int_0^1 \int_{\bar{A}} (\bar{\bar{\delta}}fu^a - \epsilon^a\bar{\bar{\delta}}\phi - \sigma^a\bar{\bar{\delta}}\epsilon - \sigma\bar{\bar{\delta}}\epsilon^a + \bar{\bar{\delta}}G)J \, d\bar{A} \, d\xi$$

$$+ \int_0^1 \int_{\bar{A}} (fu^a - \sigma\epsilon^a + G)\bar{\delta}J \, d\bar{A} \, d\xi \tag{52}$$

Various quantities for use in Eq. (45) are

$$\bar{\bar{\delta}}\phi = fL\bar{\delta}E(1-\xi)/E; \qquad \bar{\delta}J = L\bar{\delta}A + A\bar{\delta}L;$$

$$\bar{\bar{\delta}}G = -(A^{-1} + L^{-1})G \tag{53}$$

Substituting various quantities from Eqs. (47), (49), and (53) into Eq. (52), we again obtain the sensitivity expression given in Eq. (45). The sensitivity interpretation for the adjoint fields can again be easily verified.

Example 2. Cantilever Beam

This example is also discussed in Ref. 7 where sensitivity of tip deflection with respect to the length is given. Figure 4 defines the problem and the transformations to the reference volume. The design variables are chosen as $\boldsymbol{b} = (E, s, h, L)$. The tip deflection using small displacement beam theory is given as $w(L) = PL^3/3EI$ and its variation with respect to the design variables is

$$\bar{\delta}w(L) = -\frac{PL^3}{3E^2I}\bar{\delta}E - \frac{PL^3h^3}{36EI^2}\bar{\delta}s - \frac{PL^3sh^2}{12EI^2}\bar{\delta}h + \frac{PL^2}{EI}\bar{\delta}L \tag{54}$$

The functional for the tip deflection and the function G is given as

$$\psi = \int_0^1 \int_{\bar{A}} (AL)^{-1}w(\xi)\hat{\delta}(\xi-1) \, AL \, d\bar{A} \, d\xi \tag{55}$$

$$G = (AL)^{-1}w(\xi)\hat{\delta}(\xi-1); \qquad G_{,w} = (AL)^{-1}\hat{\delta}(\xi-1) \tag{56}$$

The primary and adjoint structure solutions are

$$w_{,\xi\xi} = \frac{PL^3}{EI}(1-\xi); \quad w(\xi) = \frac{PL^3\xi^2}{6EI}(3-\xi) \tag{57}$$

$$w^a_{,\xi\xi} = \frac{L^3}{EI}(1-\xi); \quad w^a(\xi) = \frac{L^3\xi^2}{6EI}(3-\xi) \tag{58}$$

The sensitivity formula of Eq. (44) is reduced to

$$\bar{\delta}\psi = \int_0^1 \int_{\bar{A}} (-\epsilon^a\bar{\bar{\delta}}\phi - \sigma\bar{\bar{\delta}}\epsilon^a - \sigma^a\bar{\bar{\delta}}\epsilon + \bar{\bar{\delta}}G)AL \, d\bar{A} \, d\xi$$

$$+ \int_0^1 \int_{\bar{A}} (-\sigma\epsilon^a + G)\bar{\delta}(AL) \, d\bar{A} \, d\xi \tag{59}$$

The following quantities are needed to complete integrations in Eq. (59):

$$\epsilon = \zeta w_{,\xi\xi}hL^{-2}; \quad \epsilon^a = \zeta w^a_{,\xi\xi}hL^{-2}; \quad \bar{\bar{\delta}}G = w\hat{\delta}(\xi-1)\bar{\delta}(AL)^{-1}$$

$$\sigma = E\epsilon; \quad \bar{\bar{\delta}}\phi = \epsilon\bar{\delta}E; \quad \sigma^a = E\epsilon^a$$

$$\bar{\bar{\delta}}\epsilon = \zeta w_{,\xi\xi}\bar{\delta}(hL^{-2}); \quad \bar{\bar{\delta}}\epsilon^a = \zeta w^a_{,\xi\xi}\bar{\delta}(hL^{-2})$$

Substituting these quantities in Eq. (59) and carrying out the integrations, we obtain the sensitivity expression given in Eq. (56). It is interesting to note that the adjoint displacement field given in Eq. (58) represents the sensitivity of the primary displacement field (Eq. 57) with respect to the load parameter P; i.e., $w^a(1) = d\psi/dP = d(PL^3/3EI)/dP$. Also, the term $(-PL^3/3E^2I)\bar{\delta}E$ is obtained directly from the second term in Eq. (44).

Example 3. Materially Nonlinear Problem

Consider the bar of Fig. 3 subjected to a load P in the x direction at the free end. The material for the bar obeys a nonlinear stress-strain law $\sigma = E\epsilon^{1/2}(\epsilon > 0)$, so $\phi = E\epsilon^{1/2}$. We will consider E, A, and L as design variables and determine the sensitivity of the tip deflection. Transformation to the reference volume gives $x = L\xi$, $a = \bar{A}A$, $J = AL$, $V = AL$, and $\bar{V} = 1$. Nonlinear analysis of the primary structure yields

$$u(\xi) = \frac{P^2L\xi}{A^2E^2}$$

$$\bar{\delta}u(L) = -\frac{2P^2L}{A^2E^3}\bar{\delta}E - \frac{2P^2L}{A^3E^2}\bar{\delta}A + \frac{P^2}{A^2E^2}\bar{\delta}L \tag{60}$$

The functional for sensitivity analysis is given as

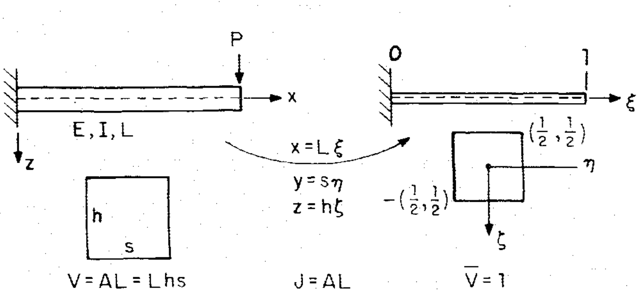

Fig. 4 Cantilever beam.

$$\psi = \int_0^1 \int_{\bar{A}} A^{-1}L^{-1}u(\xi)\hat{\delta}(\xi-1)(AL)\,\mathrm{d}\bar{A}\,\mathrm{d}\xi \tag{61}$$

$$G = A^{-1}L^{-1}u(\xi)\hat{\delta}(\xi-1); \qquad G_{,u} = A^{-1}L^{-1}\hat{\delta}(\xi-1) \tag{62}$$

The adjoint structure is linear with the stress-strain law as

$$\sigma^a = \phi_{,\epsilon}\epsilon^a = \frac{1}{2}E\epsilon^{-\frac{1}{2}}\epsilon^a = \frac{AE^2}{2P}\epsilon^a \tag{63}$$

The equilibrium equation for the adjoint structure gives

$$u^a(\xi) = \frac{2PL\xi}{A^2E^2}; \qquad u^a_{,\xi} = \frac{2PL}{A^2E^2} \tag{64}$$

The sensitivity formula of Eq. (44) reduces to

$$\bar{\delta}\psi = \int_0^1\int_{\bar{A}} (-\epsilon^a\bar{\bar{\delta}}\phi - \sigma^a\bar{\bar{\delta}}\epsilon - \sigma\bar{\bar{\delta}}\epsilon^a + \bar{\bar{\delta}}G)J\,\mathrm{d}\bar{A}\,\mathrm{d}\xi$$
$$+ \int_0^1\int_{\bar{A}} (-\sigma\epsilon^a + G)\bar{\delta}J\,\mathrm{d}a\,\mathrm{d}\xi \tag{65}$$

Various quantities for Eq. (65) are

$$\epsilon = u_{,\xi}L^{-1} = \frac{P^2L}{A^2E^2}; \qquad \epsilon^a = u^a_{,\xi}L^{-1} = \frac{2PL}{A^2E^2}$$

$$\bar{\bar{\delta}}G = u\hat{\delta}(\xi-1)\bar{\delta}(AL)^{-1}$$

$$\sigma = E\epsilon^{\frac{1}{2}}; \qquad \phi_{,\epsilon} = (\tfrac{1}{2})E\epsilon^{-\frac{1}{2}},\ \bar{\bar{\delta}}\phi = \epsilon^{\frac{1}{2}}\bar{\delta}E; \qquad \sigma^a = \phi_{,\epsilon}\epsilon^a$$

$$\bar{\bar{\delta}}\epsilon = u_{,\xi}\bar{\delta}L^{-1} = -L^{-2}u_{,\xi}\,\bar{\delta}L; \qquad \bar{\bar{\delta}}\epsilon^a = -L^{-2}u^a_{,\xi}\bar{\delta}L$$

Substituting these quantities into Eq. (65), we obtain the sensitivity formula of Eq. (60). It is interesting to note the sensitivity meaning of the adjoint displacement field in Eq. (64); i.e., $u^a(1) = \mathrm{d}\psi/\mathrm{d}P$. Also, the term $(-2P^2L/A^2E^3)\bar{\delta}E$ is obtained directly from the second term in Eq. (44).

Example 4. Geometrically Nonlinear Problem

Consider the two-bar structure shown in Fig. 5. The material for the structure is linear, so $\sigma = E\epsilon$. A geometrically nonlinear analysis for the problem is presented in Refs. 26 and 27. The approximations used there are also used in the present paper in order to perform analytical design sensitivity analysis. Transformation to the reference volume is shown in the figure. The design variables for the problem are $\boldsymbol{b} = (E, A, L)$. The strain for the problem is given as

$${}^t_0\epsilon = ({}^tL - {}^0L)/{}^0L = (1 + w^2L^{-2})^{\frac{1}{2}} - 1$$
$$= 1 + \tfrac{1}{2}w^2L^{-2} + \cdots - 1 \doteq \tfrac{1}{2}w^2L^{-2} \equiv \epsilon \tag{66}$$

where the terms higher than order two are neglected. This, in fact, is the Green-Lagrange strain measure that is given as

$$\epsilon = \frac{1}{2}\left(\frac{{}^tL^2 - {}^0L^2}{{}^0L^2}\right) = \frac{1}{2}\left(\frac{{}^0L^2 + w^2 - {}^0L^2}{{}^0L^2}\right) = \frac{1}{2}w^2L^{-2}$$

The deflection at the center and member strains are calculated as

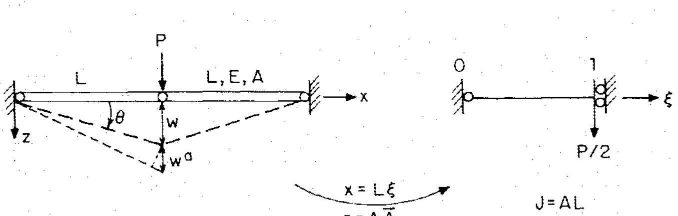

Fig. 5 Two-member structure.

$$w = \frac{P^{\frac{1}{3}}L}{(EA)^{\frac{1}{3}}}; \qquad \bar{\epsilon} = \epsilon L^2 = \frac{1}{2}w^2 = \frac{P^{\frac{2}{3}}L^2}{2(EA)^{\frac{2}{3}}} \tag{67}$$

The incremental equilibrium equation in terms of displacement at the center is $3EAw^2L^{-3}\delta w = \delta P$. The functional for sensitivity analysis is given as

$$\psi = \int_0^1\int_{\bar{A}} (AL)^{-1}w(\xi)\hat{\delta}(\xi-1)AL\,\mathrm{d}\bar{A}\,\mathrm{d}\xi \tag{68}$$

$$G = (AL)^{-1}w(\xi)\hat{\delta}(\xi-1); \qquad G_{,w} = (AL)^{-1}\hat{\delta}(\xi-1) \tag{69}$$

The equilibrium equation for the adjoint structure (using the incremental equilibrium equation of the primary structure) is

$$3EAL^{-3}w^2w^a = \int_0^1\int_{\bar{A}} (AL)^{-1}\hat{\delta}(\xi-1)AL\,\mathrm{d}\bar{A}\,\mathrm{d}\xi = 1$$

$$w^a = \frac{L^3}{3EAw^2} = \frac{L}{3E^{\frac{1}{3}}A^{\frac{1}{3}}P^{\frac{2}{3}}} \tag{70}$$

Total axial displacement and displacement at any point are given as

$$u^a(L) = w^a\sin\theta = w^awL^{-1}; \qquad u^a(\xi) = w^awL^{-1}\xi \tag{71}$$

The adjoint strains are given as

$$\epsilon^a = w^awL^{-2} = \bar{\epsilon}^aL^{-2},\ \bar{\epsilon}^a = w^aw \tag{72}$$

The sensitivity formula of Eq. (44) reduces to

$$\bar{\delta}\psi = 2\int_0^1\int_{\bar{A}} (-\epsilon^a\bar{\bar{\delta}}\phi - \sigma^a\bar{\bar{\delta}}\epsilon - \sigma\bar{\bar{\delta}}\epsilon^a + \bar{\bar{\delta}}G)J\,\mathrm{d}\bar{A}\,\mathrm{d}\xi$$
$$+ 2\int_0^1\int_{\bar{A}} (-\sigma\epsilon^a + G)\bar{\delta}J\,\mathrm{d}\bar{A}\,\mathrm{d}\xi \tag{73}$$

Note that a factor of 2 is used because volume integrals in Eq. (44) are for the entire structure. Various quantities for Eq. (73) are

$$\bar{\bar{\delta}}\phi = \epsilon\bar{\delta}E; \ \bar{\bar{\delta}}\epsilon = \bar{\epsilon}\bar{\delta}L^{-2} = -w^2L^{-3}\bar{\delta}L$$

$$\bar{\bar{\delta}}\epsilon^a = \bar{\epsilon}^a\bar{\delta}L^{-2} = -2ww^aL^{-3}\bar{\delta}L$$

$$\sigma = E\epsilon = \tfrac{1}{2}Ew^2L^{-2}; \ \sigma^a = E\epsilon^a = Eww^aL^{-2}$$

$$\bar{\delta}J = L\bar{\delta}A + A\bar{\delta}L \tag{74}$$

Substituting these quantities into Eq. (73), we get

$$\bar{\delta}\psi = -\frac{P^{\frac{1}{3}}L}{3E^{4/3}A^{\frac{1}{3}}}\bar{\delta}E - \frac{P^{\frac{1}{3}}L}{3E^{\frac{1}{3}}A^{4/3}}\bar{\delta}A + (P/EA)^{\frac{1}{3}}\bar{\delta}L \tag{75}$$

which can also be obtained directly from Eq. (67). Comparing w and w^a in Eqs. (67) and (70), we again observe the sensitivity interpretation of the adjoint displacement field.

Next, consider the member stress given in Eq. (74) as $\sigma = (P^{\frac{2}{3}}E^{\frac{1}{3}})/(2A^{\frac{2}{3}})$. Its design sensitivity is given as

$$\bar{\delta}\sigma = \frac{P^{\frac{2}{3}}}{6E^{\frac{2}{3}}A^{\frac{2}{3}}}\bar{\delta}E - \frac{P^{\frac{2}{3}}E^{\frac{1}{3}}}{3A^{5/3}}\bar{\delta}A + (0)\bar{\delta}L \tag{76}$$

The functional for design sensitivity analysis is given as

$$\psi = \int_0^1\int_{\bar{A}} (AL)^{-1}\sigma AL\,\mathrm{d}\bar{A}\,\mathrm{d}\xi = \int_0^1\int_{\bar{A}} G(\sigma)AL\,\mathrm{d}\bar{A}\,\mathrm{d}\xi \tag{77}$$

The adjoint load $G_{,u}$ in this case is zero, but initial strain in the adjoint structure and stress-strain law are given as $\epsilon^{ai} = G_{,\sigma} = (AL)^{-1}$, $\sigma^a = E(\epsilon^a - \epsilon^{ai})$. The adjoint equilibrium equation in terms of central displacement gives $w^a = (wA)^{-1}L/3$ and $\epsilon^a = ww^aL^{-2} = (AL)^{-1}/3$. If we substitute appropriate quantities in Eq. (44), it can be verified that Eq. (76) is obtained. It can be also directly verified that $w^a = d\psi/dP \equiv d\sigma/dP$.

VII. Concluding Remarks

A general formula for the design sensitivity analysis of linear and nonlinear structures using the variational approach has been developed. Equations of continuum mechanics are used and the concepts of reference volume and adjoint structure are exploited. Use of the formula is demonstrated on a few simple analytical problems. The formula has also been successfully applied to simple large strain and time-dependent material property (Voigt model) problems.

The theory can be adapted for the finite-element modeling of structures. The finite-element models for the primary and adjoint structures can be independent of each other to adapt the method for use with existing finite-element analysis codes. For the modeling of design optimization problems, the concept of a reference volume is translated into the concept of a design element that is invariant with respect to design changes. These observations can have considerable implications in numerical implementations for design sensitivity analysis and optimization of complex structures.

The design sensitivity analysis of nonlinear stresses, strains, displacements, and the buckling load, starting with a finite-element model of the structure, has been also recently developed.[28-34] It has been shown that the optimal designs with nonlinear response can be substantially different from those with the linear response. It can be shown that the continuum approach presented in this paper is equivalent to the discrete approach of Refs. 28-33 in numerical implementations. However, the continuum approach offers some insights that are not apparent if one starts with a discretized model. For example, design sensitivity analysis with discretized models needs only adjoint displacements because stresses and strains are expressed in terms of displacements. The adjoint stresses and strains are neither needed nor calculated. The continuum approach uses adjoint stresses and strains also. In addition, the sensitivity interpretation of adjoint displacements, stresses, and strains is readily available. With the discretized approach, the sensitivity interpretation of the adjoint displacements had to be discovered using the Lagrangian approach.[21,22]

The issues of equivalence between continuum and discrete approaches, and design sensitivity analysis with time-dependent effects and reverse plasticity, will be discussed in subsequent publications.

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